Analytical Solutions for the *n*th Derivatives of Eigenvalues and Eigenvectors for a Nonlinear Eigenvalue Problem

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Abstract

THE nonlinear eigenvalue problem discussed here is defined by the following equations:

$$A(\pi,\lambda)\, x = 0 \qquad (1)$$

$$x^{\dagger} K(\pi,\lambda)\, x = 1 \qquad (2)$$

where $A(\pi,\lambda)$ is an $n \times n$ matrix† representing a nonlinear operator on the unrepeated eigenvalue λ; x is the corresponding $1 \times n$ eigenvector that is K-normalized by an $n \times n$ positive definite hermitian, K; $x^{\dagger}$ is the complex conjugate transpose of x; and π is a real scalar parameter. The eigenvalue λ is obtained by setting the determinant of A to zero and solving the resulting characteristic equation for $\lambda(\pi)$. The corresponding eigenvector, x, is then calculated from Eqs. (1) and (2). It is assumed that λ is a differentiable function of π. Elements of A and K, a_{ij} and k_{ij} ($1 = 1, \ldots, n$; $j = 1, \ldots, n$) must also be differentiable functions of π and are, in general, nonlinear functions of λ and π.

The objective of this paper is to find explicit analytical solutions for the nth derivative of the unrepeated eigenvalue λ, $\lambda^{(n)}$, and the nth derivative of the corresponding eigenvector x, $x^{(n)}$, with respect to π. Various numerical techniques have been proposed in the past[1] but no analytical solutions of this problem have been available.

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The differentiation of Eq. (1) with respect to π, with the subscripts π and Π indicating a partial and a total derivative, respectively, gives

$$A_{\Pi}\, x + A x_{\Pi} = 0 \qquad (3)$$

or

$$A_{\lambda}\, x \lambda_{\Pi} + A x_{\Pi} = -A_{\Pi}\, x \qquad (4)$$

since

$$A_{\Pi} = A_{\pi} + A_{\lambda}\lambda_{\Pi} \qquad (5)$$

Similarly, the differentiation of Eq. (2), with respect to π and using

$$K_{\Pi} = K_{\pi} + \lambda_{\Pi} K_{\lambda} \qquad (6)$$

gives

$$x^{\dagger} K_{\lambda} x \lambda_{\Pi} + 2x^{\dagger} K x_{\Pi} = -x^{\dagger} K_{\pi} x \qquad (7)$$

Equations (4) and (7) form the following system of linear algebraic equations, with λ_{Π} and x_{Π} unknown:

$$\begin{bmatrix} A & A_{\lambda}x \\ 2x^{\dagger}K & \alpha \end{bmatrix} \begin{pmatrix} x_{\Pi} \\ \lambda_{\Pi} \end{pmatrix} = \begin{pmatrix} -A_{\pi}x \\ -\beta \end{pmatrix} \qquad (8)$$

where

$$\alpha = x^{\dagger} K_{\lambda} x \qquad (9)$$

$$\beta = x^{\dagger} K_{\pi} x \qquad (10)$$

The systems of Eq. (8) were obtained, and solved numerically in Ref. 2. The solution of Eq. (8) is written as

$$\begin{pmatrix} x_{\Pi} \\ \lambda_{\Pi} \end{pmatrix} = \begin{bmatrix} A & A_{\lambda}x \\ 2x^{\dagger}K & \alpha \end{bmatrix}^{-1} \begin{pmatrix} -A_{\pi}x \\ -\beta \end{pmatrix} \qquad (11)$$

Two distinct cases, $\alpha = 0$ and $\alpha \neq 0$, are considered. The solutions for $\alpha = 0$ are

$$2x^{\dagger} K\, x_{\Pi} = -\beta \qquad (12)$$

$$x_{\Pi} = -\frac{\beta}{2} x \qquad (13)$$

The solution for $\alpha \neq 0$ is given as

$$\begin{pmatrix} x_{\Pi} \\ \lambda_{\Pi} \end{pmatrix} = \begin{bmatrix} P & x/2 \\ -2x^{\dagger}K\,P/\alpha & 0 \end{bmatrix} \begin{pmatrix} -A_{\pi}x \\ -\beta \end{pmatrix} \qquad (14)$$

or

$$x_{\Pi} = -P A_{\pi} x - \frac{\beta}{2} x \qquad (15)$$

and

$$\lambda_{\Pi} = 2\, x^{\dagger} \left(K \frac{P A_{\pi}}{\alpha}\right) x \qquad (16)$$

Theorems for both $\alpha = 0$ and $\alpha \neq 0$ are stated here without proofs, which are given in the unabridged version of this paper.

Theorem I: For the nonlinear eigenvalue problem previously defined, and for $\alpha = 0$

$$-(A_{\lambda}^{-1}A_{\pi})^{(n)}x = \lambda^{(n+1)}x$$

The superscripts n and $n + 1$ indicate the nth and $n + 1$st derivatives with respect to the parameter π.

Theorem II: For the nonlinear eigenvalue problem previously defined, and for $\alpha = 0$, the nth derivative of the eigenvalue λ with respect to the parameter π is

$$\lambda^{(n)} = -x^{\dagger} K (A_{\lambda}^{-1}A_{\pi})^{(n-1)} x$$

Theorem III: For the nonlinear eigenvalue problems previously defined, and for $\alpha = 0$, the nth derivative of the eigenvector x is

$$x^{(n)} = -\frac{1}{2}(\beta x)^{(n-1)}$$

Theorem IV: For the nonlinear eigenvalue problem previously defined, and for $\alpha = 0$

$$A^{(n-k)} x^{(k)} = 0, \quad k = 0, 1, \ldots, n$$

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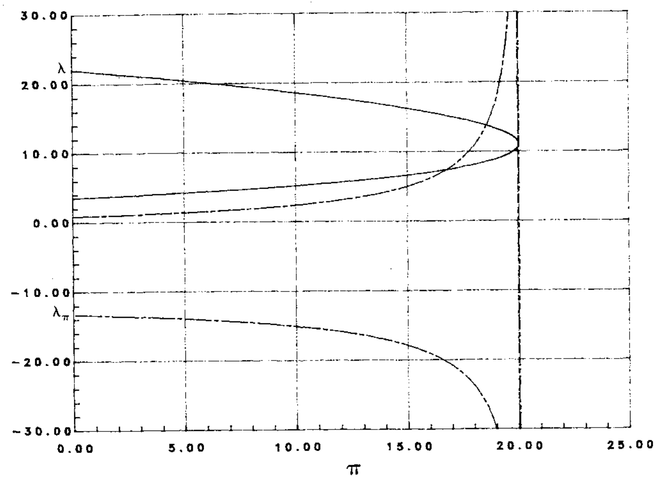

Fig. 1 λ and λ_π.

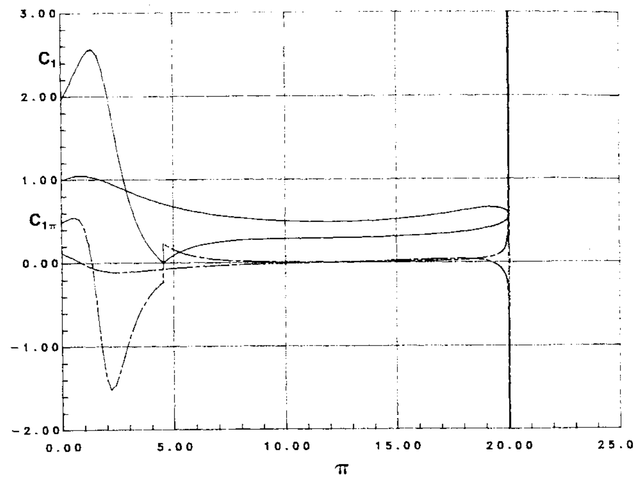

Fig. 2 C_1 and $C_{1\pi}$.

Theorem V: For the nonlinear eigenvalue problem previously defined, and for $\alpha \neq 0$, the nth derivative of the eigenvalue λ with respect to the parameter π is

$$\lambda^{(n)} = 2\left[x^\dagger K\left(\frac{PA_\pi}{\alpha}\right)x\right]^{(n-1)}$$

Theorem VI: For the nonlinear eigenvalue problem previously defined, the nth derivative of the eigenvalue $x^{(n)}$ with respect to the parameter π is

$$x^{(n)} = -(PA\,x)_\pi^{(n-1)} - \frac{(\beta x)^{(n-1)}}{2}$$

These theorems are also valid for the linear eigenvalue problem.

The theory outlined here applies to many problems in dynamics, and other branches of applied mathematics such as the theories of oscillation, elasticity, and optimization. Its application to an elementary, nonconservative problem of the theory of elasticity, namely the problem of the oscillating bar compressed by a tangential force, follows. In Ref. 3, the matrix A is given by

$$\begin{aligned}
a_{11} &= a_{13} = 0\\
a_{22} &= a_{24} = 0\\
a_{33} &= a_{34} = 0\\
a_{43} &= a_{44} = 0\\
a_{12} &= a_{14} = 1\\
a_{21} &= \hat\beta\\
a_{23} &= \hat\alpha\\
a_{31} &= \hat\beta^2\sin\hat\beta + \hat\alpha\hat\beta\,\sinh\hat\alpha\\
a_{32} &= \hat\beta^2\cos\hat\beta + \hat\alpha^2\cosh\hat\alpha\\
a_{41} &= -(\hat\beta^3\cos\hat\beta + \hat\beta\hat\alpha^2\cosh\hat\alpha\\
a_{42} &= \hat\beta^3\sin\hat\beta - \hat\alpha^3\sinh\hat\alpha
\end{aligned} \tag{17}$$

which was obtained by substituting the solution $V(\xi) = c_1 \sin\hat\beta\xi + c_2\cos\hat\beta\xi + c_3\sinh\hat\alpha\xi + c_4\cosh\hat\alpha\xi$ into the governing equation

$$\frac{d^4V}{d\xi^4} + \pi\frac{d^2V}{d\xi^2} - \omega^2 V = 0, \quad \xi = \frac{Z}{\ell} \tag{18}$$

and by using the appropriate boundary conditions. In Eq. (18), ω is a nondimensional vibration frequency, and the nondimensional parameter π is proportional to the tangential force P, is the square of the bar's length ℓ, and is inversely proportional to the bending stiffness EJ.

That is

$$\pi = \frac{P\ell^2}{EJ} \tag{19}$$

Also, $\hat\alpha$ and $\hat\beta$ are defined as

$$\hat\alpha = \sqrt{(\pi/4 + \lambda)\ 1/2 - \pi/2^{1/2}} \tag{20a}$$

$$\hat\beta = \sqrt{(\pi/4 + \lambda)\ 1/2 - \pi/2^{1/2}} \tag{20b}$$

where $\lambda \equiv \omega^2$ and ω was calculated from det $A = 0$ as a function of π. The normality condition

$$\int_0^1 V^2 d\xi = 1 \tag{21}$$

can be written as

$$x^\dagger K\,x = 1 \tag{22}$$

where the elements of the matrix K are evaluated in closed form and the eigenvector is given by $x^\dagger = (c_1\, c_2\, c_3\, c_4)$. The critical value of the parameter π, π^*, which gives the critical frequency, is calculated from Eq. (16) to be $\pi^* = 20.05059$ .... The derivatives of λ and c_1 are shown in Figs. 1 and 2.

The theory presented here was used to calculate the first and second derivatives of the solar array's eigenvalues and eigenvectors for the Hermes satellite. The problem is quite complex and is discussed in Ref. 4, and in detail in Ref. 2.

Conclusions

This paper analytically calculates the nth derivatives of eigenvalues and eigenvectors for a nonlinear eigenvalue problem. It also states the related theorems. Some applications are mentioned and an example is discussed.

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